Toward a unified model of sovereign quanto CDS spreads and government bond yields

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Abstract

A unified model that consistently evaluates sovereign quanto credit default swap (CDS) and government bonds is developed. By product, a new procedure is proposed to calibrate stochastic processes of the risk-free interest rate and the sovereign default intensity to sovereign quanto CDS spreads and government bond yields. Fractional step methods are applied to solve partial differential equations for CDS spreads and bond yields, which cannot be solved using a standard finite difference method due to a cross derivative term. An empirical study is conducted on United States, German and Portuguese quanto CDS spreads during the European sovereign debt crisis. The stochastic processes of the risk-free interest rates in USD and Euros and the default intensities of United States, German and Portuguese are simultaneously estimated and reveals that sovereign quanto CDS spread differentials are partially explained by introducing a correlation between the risk-free interest rate and the sovereign default intensity. Numerical analysis shows that the larger correlation between them leads to the smaller CDS spread.

Keywords : sovereign quanto credit default swap (CDS) spreads; government bond yields; default intensity; risk-free interest rates; correlation; fractional step methods

JEL classifications : G12; G13; G15

1 Introduction

Credit default swaps (CDSs) are popular credit derivatives that enable investors to hedge credit risk. Investors who take a long position in corporate bonds can hedge against an associated credit risk by purchasing a corporate CDS whose reference entity is the same as the issuer of the corporate bond. The underlying credit risk for both corporate bonds and corporate CDS is common, and thus, corporate CDS and corporate bonds can be jointly priced using credit risk and risk-free interest rate models. This argument also applies to evaluate sovereign CDSs and government bonds. Therefore, a unified model for the risk-free interest rate and the default intensity can jointly price sovereign CDS and government bonds.

Practitioners and academic researchers assume that interest rates estimated from the U.S. Treasury or German government bonds are risk-free interest rates. In contrast, the U.S. and German CDS spreads are not zero, and this fact implies that bond yields for the U.S. and Germany comprise the risk-free interest rates and the credit risk spreads. Then, the 'genuine' risk-free interest rates should be derived by subtracting credit risk spreads from government bond yields. Credit risk for a government can be estimated from sovereign CDS spreads. Moreover, CDS involves a sequence of premium payments and a possible protection repayment, and therefore, CDS spreads depend on the risk-free interest rate and the credit risk spread. Thus, estimation of the risk-free interest rate and the default intensity is complicated.

Few studies have proposed procedures to estimate the 'genuine' risk-free interest rate which is immune to credit risk spreads. Kagraoka and Moussa (2014) estimated the risk-free interest rate in Japanese yen (JPY) and the default intensity for Japan using Japanese CDS spreads quoted in JPY and Japanese government bond yields. They assumed a deterministic

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risk-free interest rate and a constant default intensity, and they did not consider the correlation between the risk-free interest rates and the default intensity for Japan. Kagraoka (2019) facilitated sovereign CDS spreads quoted in a foreign currency to estimate the stochastic processes for the risk-free interest rate and the sovereign default intensity; however, he assumed zero correlation between the risk-free interest rate and the default intensity.

Several studies investigate the correlation between the risk-free interest rate and the sovereign default intensity. Brigo and Alfonsi (2005) applied the shifted CIR model to the risk-free interest rate and the default intensity. They concluded that the impact of the correlation between the risk-free interest rate and the default intensity was small. It is suspected that the correlation might be neglected because Brigo and Alfonsi (2005) calibrated their model to market data on October 25, 2002, where CDS spreads and their volatility were quite low, and this market characteristic made the effect of correlation negligible. Chen et al. (2008) evaluated corporate CDS using multifactor risk-free interest rates and default intensity. They estimated the correlation between the risk-free interest rate and the default intensity; however, they did not investigate the effect of the correlation on CDS spreads.

CDS spreads are quoted in various currencies. In general, CDS spreads quoted in foreign currencies are higher than those quoted in a domestic currency. For example, U.S. CDS spreads quoted in Euros are higher than those quoted in US dollars (USD). Likewise, German CDS spreads quoted in USD are higher than those quoted in Euros. Quanto CDS spread differentials are differences in CDS spreads for the same reference entity among different quoted currencies. In addition to credit risk, several factors might produce price discrepancies between sovereign CDS spreads. Elizalde et al. (2010) pointed out that quanto CDS spreads were affected by spread and FX volatilities, their correlation, and the potential FX depreciation on default. Arce et al. (2013) documented that the deviations between CDS spreads and bond yields were related to counterparty risk, common volatility in EMU equity markets, market illiquidity, funding costs, flight-to-quality, and the volume of debt purchases by the European Central Bank (ECB) in the secondary market. Fontana and Scheicher (2016) pointed out that sovereign CDS and government bonds were priced, violating the no-arbitrage condition due to short-selling frictions and funding frictions. Augustin et al. (2020) and Brigo et al. (2019) proposed a model in which currency devaluation in conjunction with default events caused quanto CDS spread differentials. Brigo et al. (2019) considered the correlation between the default intensities and FX rate and a conditional devaluation jump of the FX rate upon default to explain the quanto CDS spread differences. If the depreciation of the FX rate at the default might influence the value of CDS protection, this effect would affect both premium payment legs and a protection leg, and the effect of the FX depreciation on both sides of legs would offset. Then, the CDS spread is immune to the FX rate depreciation.

The purpose of this study is fourfold. First, this study presents a unified model to jointly evaluate sovereign CDSs and government bonds and proposes a calibration procedure for the unified model. Regarding government bonds as risky assets, stochastic processes for the risk-free interest rate and the default intensity are constructed. Specifically, the Vasicek model for the risk-free interest rate and the exponentialized Vasicek model for the default intensity are chosen to illustrate the unified model for the analytical tractability; however, other stochastic processes might be appropriate to obtain a closer fit to the market data. Second, this study proposes a calibration procedure of the stochastic processes for the risk-free interest rate and the default intensity and examines whether the chosen stochastic models fit to the market data. Third, this study investigates the impact of the correlation between the risk-free interest rates and the default intensity on CDS spreads. Thus, this paper investigates whether the correlation between the risk-free interest rates and default intensity can explain the price differences observed in quanto CDS spreads. Forth, fractional step methods developed by Yanenko (1971) are applied to solve the partial differential equations (PDEs) for sovereign CDS spreads and government bond prices, which cannot be solved using standard finite difference methods due to a cross derivative term.

The organization of this paper is as follows. The next section explains a pricing model for government bond yields and sovereign CDS spreads and presents PDEs that are satisfied by defaultable bond prices and premium payment and protection legs of CDS. Then, a numerical method, known as the fractional step method, is reviewed to solve the PDEs. Next, a calibration procedure is presented, and subsequently, calibration results are provided. The final section concludes this work.

2 A unified model

Market participants presume that government bond yields in the U.S. or Germany is a risk-free interest rate; however, these yields comprise the risk-free interest rate and credit spread because CDS spreads for the U.S. and Germany deviate from zero. Then, the risk-free interest rate for USD (resp. Euros) and default intensity for the U.S. (resp. Germany) should be distilled from government bond yields for the U.S. (resp. German) Treasuries and sovereign CDS spreads for the U.S. (resp. Germany). To do so, a reduced-form model is constructed. The stochastic processes for the risk-free interest rate and the default intensity are modelled and applied to evaluate the government bond yields and the sovereign CDS spreads. A derivative price is obtained as a solution to a PDE with respect to the risk-free interest rate and the default intensity.

2.1 Stochastic processes for the risk-free interest rate and the default intensity

This subsection constructs a stochastic process for the risk-free interest rate in USD (resp. Euros) and the default intensity for the U.S. (resp. Germany). Appropriate stochastic processes for the risk-free interest rate and the default intensity can reproduce the theoretical prices fitted to the corresponding market prices at an instant of time. Thus, the stochastic processes solely under a martingale measure are constructed, and the stochastic processes under a physical measure are disregarded.[1]

Theoretically, the risk-free interest rate is assumed to be positive; however, market participants have witnessed negative government bond yields in European countries and Japan. This study employs a risk-free interest rate model that allows negative rates. Thus, the risk-free interest rate $r_t = x_t$ is assumed to follow the Vasicek model (see, for example, Brigo and Mercurio, 2006),

$$dx_t = \kappa_x(\theta_x - x_t)\, dt + \sigma_x\, dW_{x,t}. \tag{1}$$

Default is modelled using the reduced-form approach developed by Lando (1998). Let τ be a nonnegative random variable, called a default time, on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$. A time-$t$ survival probability that the default time exceeds T, $S(t,T)$, is expressed as

$$S(t,T) = \Pr\left[\tau > T \middle| \mathcal{F}_t\right] = \mathrm{E}\left[\exp\left(-\int_t^T \lambda_s\, ds\right)\middle| \mathcal{F}_t\right], \tag{2}$$

where the default intensity, λ_t, is a nonnegative function of t and is modelled by a stochastic process. Many stochastic processes originally proposed as the risk-free interest rate are applied to model the default intensity (see, for example, Brigo and Mercurio, 2006); a square root process proposed by Cox et al. (1985) (CIR) and a log-normal model of Black and Karasinski (1991) are popular among them. Pan and Singleton (2008) documented that the log-normal processes captured most of the variation in the term structure of CDS spreads. Therefore, the default intensity is modelled as an exponential Vasicek type, which is equivalent to the Black and Karasinski model, and its stochastic process under $(\Omega, \mathcal{F}, \mathbb{Q})$ is written as,

$$\lambda_t = e^{y_t}, \tag{3}$$

$$dy_t = \kappa_y\left(\theta_y - y_t\right) dt + \sigma_y\, dW_{y,t}. \tag{4}$$

Quadratic variation of $dW_{x,t}$ and $dW_{y,t}$ is

$$d\langle W_x, W_y \rangle_t = \rho\, dt. \tag{5}$$

The time-t derivative price, f_t, satisfies the following PDE),

[1] Stochastic processes can be estimated under both martingale measure and physical measure by applying the likelihood maximization proposed by Duan (1994). This study considers the stochastic processes only under the martingale measure because the likelihood maximization suffers a large computational burden due to the complex numerical calculation of bond yields and CDS spreads.

$$\frac{\partial f_t}{\partial t} + \frac{1}{2}\sigma_x^2 \frac{\partial^2 f_t}{\partial x_t^2} + \rho\sigma_x\sigma_y \frac{\partial^2 f_t}{\partial x_t \partial y_t} + \frac{1}{2}\sigma_y^2 \frac{\partial^2 f_t}{\partial y_t^2} + \kappa_x(\theta_x - x_t)\frac{\partial f_t}{\partial x_t} + \kappa_y(\theta_y - y_t)\frac{\partial f_t}{\partial y_t} - x_t f_t - e^{y_t} f_t + e^{y_t} g_t = 0, \tag{6}$$

where g_t is a cashflow at the default time.

2.2 Evaluation of defaultable bonds

First, a defaultable bond with zero recovery is considered. The time-t price of bond maturing at time T is given as

$$(\text{Principal of bond}) = \mathrm{E}_t\left[\exp\left(-\int_t^T r_s\, ds\right) 1_{\{\tau > T\}}\right]. \tag{7}$$

This price is also obtained by solving the pricing PDE, eq. (6), with a terminal condition, $f_T(x_T, y_T) = 1$. Recovery of bond is given by

$$(\text{Recovery of bond}) = \delta \mathrm{E}_t\left[\exp\left(-\int_t^\tau r_s\, ds\right) 1_{\{\tau \le T\}}\right], \tag{8}$$

where recovery rate is denoted by δ. The value of recovery is also obtained by solving the following pricing PDE:

$$\frac{\partial f_t}{\partial t} + \frac{1}{2}\sigma_x^2 \frac{\partial^2 f_t}{\partial x_t^2} + \rho\sigma_x\sigma_y \frac{\partial^2 f_t}{\partial x_t \partial y_t} + \frac{1}{2}\sigma_y^2 \frac{\partial^2 f_t}{\partial y_t^2} + \kappa_x(\theta_x - x_t)\frac{\partial f_t}{\partial x_t} + \kappa_y(\theta_y - y_t)\frac{\partial f_t}{\partial y_t} - x_t f_t - e^{y_t} f_t + \delta e^{y_t} = 0, \tag{9}$$

where the last term in the LHS corresponds to a recovery repayment with recovery rate δ and conditional default probability $\lambda_t = e^{y_t}$. The defaultable bond price is a sum of the principal and the recovery.

2.3 Evaluation of CDS

The CDS premium is periodically paid; in a standard contract on CDS, premium payments occur quarterly. Periodic premium payments are interpreted as a portfolio of defaultable bonds with face value equal to a quarter of CDS spreads. The value of periodic premium payments is given by

$$(\text{Periodic premium payments of CDS}) = \frac{1}{4}\sum_{i=1}^{n} \mathrm{E}_t\left[\exp\left(-\int_t^{T_i} r_s\, ds\right) 1_{\{\tau > t_i\}}\right], \tag{10}$$

where premium payments are made quarterly at premium payment date $T_i\ (i = 1, \ldots, n)$ with $T_i - T_{i-1} = \frac{1}{4}$. A CDS buyer pays accrued payments when the reference entity defaults during premium payment dates,

$$(\text{Accrued premium payment of CDS}) = \mathrm{E}_t\left[\alpha(T(\tau), \tau)\exp\left(-\int_t^\tau r_s\, ds\right) 1_{\{\tau \le T_n\}}\right], \tag{11}$$

where $\alpha(t, S)$ is the year fraction between times t and S, and $T(S) = \max_{i=0,\ldots n}(T_i : T_i < S)$ with $T_0 = t$. The value of accrued premium payment is also obtained by solving the following pricing PDE:

$$\frac{\partial f_t}{\partial t} + \frac{1}{2}\sigma_x^2 \frac{\partial^2 f_t}{\partial x_t^2} + \rho\sigma_x\sigma_y \frac{\partial^2 f_t}{\partial x_t \partial y_t} + \frac{1}{2}\sigma_y^2 \frac{\partial^2 f_t}{\partial y_t^2} + \kappa_x(\theta_x - x_t)\frac{\partial f_t}{\partial x_t} + \kappa_y(\theta_y - y_t)\frac{\partial f_t}{\partial y_t} - x_t f_t - e^{y_t} f_t + \alpha(T(t), t)e^{y_t} = 0. \tag{12}$$

where the last term in the LHS corresponds to an accrued premium payment with the conditional default probability $\lambda_t = e^{y_t}$. The value of protection repayment is given by

$$(\text{Protection repayment of CDS}) = (1-\delta)\mathrm{E}_t\left[\exp\left(-\int_t^\tau r_s\, ds\right) 1_{\{\tau \le T_n\}}\right]. \tag{13}$$

where $1-\delta$ is a loss given default. The value of protection repayment is obtained by solving the following PDE:

$$\frac{\partial f_t}{\partial t} + \frac{1}{2}\sigma_x^2 \frac{\partial^2 f_t}{\partial x_t^2} + \rho\sigma_x\sigma_y \frac{\partial^2 f_t}{\partial x_t \partial y_t} + \frac{1}{2}\sigma_y^2 \frac{\partial^2 f_t}{\partial y_t^2} + \kappa_x(\theta_x - x_t)\frac{\partial f_t}{\partial x_t} + \kappa_y(\theta_y - y_t)\frac{\partial f_t}{\partial y_t} - x_t f_t - e^{y_t} f_t + (1-\delta)e^{y_t} = 0. \tag{14}$$

where the last term in the LHS corresponds to loss given default with recovery rate δ and conditional default probability $\lambda_t = e^{y_t}$. The CDS spread is set to a level at which the sum of the periodic and accrued payments equates to the protection repayment,

$$(\text{CDS spread}) = \frac{(\text{the protection repayment})}{(\text{the periodic premium payment}) + (\text{the accrued premium payment})}. \tag{15}$$

All PDEs have cross derivative terms. These PDEs are solved numerically by applying fractional step methods.

3 The fractional step methods

The PDEs in eqs. (9), (12) and (14), do not have analytical solutions; thus, this study resorts to a numerical method. The PDEs involve two space dimensions, x_t and y_t, and they have cross derivative terms. In general, the alternating direction implicit (ADI) method is applied to solve the PDEs in multiple dimensions; however, Duffy (2006) noted that the original ADI method is not applicable to solve the PDE having a cross derivative term. The fractional step method developed by Yanenko (1971) is a variation of the finite difference methods and can be applied to solve the PDE with a cross derivative term. Therefore, in this study, the fractional step method is employed. In the following, the fractional step methods are reviewed.

First, the finite difference method is explained. A PDE in two space dimensions is written in the following general representation:

$$\frac{\partial u}{\partial t} = A_{xx}\frac{\partial^2 u}{\partial x^2} + 2A_{xy}\frac{\partial^2 u}{\partial x \partial y} + A_{yy}\frac{\partial^2 u}{\partial y^2} + B_x\frac{\partial u}{\partial x} + B_y\frac{\partial u}{\partial y} + C_x u + C_y u + D_x + D_y, \tag{16}$$

$$0 \le t \le T, \ x_L \le x \le x_U, \text{ and } y_L \le y \le y_U, \tag{17}$$

with an initial condition,

$$u(0,x,y) = g(x,y), \tag{18}$$

and boundary conditions,

$$\lim_{x\to x_L}\frac{\partial^2 u}{\partial x^2}(t,x,y) = 0, \quad \lim_{x\to x_U}\frac{\partial^2 u}{\partial x^2}(t,x,y) = 0, \tag{19}$$

$$\lim_{y\to y_L}\frac{\partial^2 u}{\partial y^2}(t,x,y) = 0, \quad \lim_{y\to y_U}\frac{\partial^2 u}{\partial y^2}(t,x,y) = 0. \tag{20}$$

Transform the PDE to finite difference relations by discretizing the time axis and space axis as

$$t_n = n(\Delta t), \quad n = 0,1,\ldots,N, \quad \Delta t = \frac{T}{N}, \tag{21}$$

$$x_i = x_0 + i(\Delta x), \quad i = 0,1,\ldots,M, \quad \Delta x = \frac{x_U - x_L}{M}, \tag{22}$$

$$y_j = y_0 + j(\Delta y), \quad j = 0,1,\ldots,M, \quad \Delta y = \frac{y_U - y_L}{M}. \tag{23}$$

Derivative terms are expressed using a central difference scheme,

$$\frac{\partial u}{\partial x}(t_n,x_i,y_j) = \frac{1}{2}\frac{u(t_n,x_{i+1},y_j) - u(t_n,x_{i-1},y_j)}{\Delta x}, \tag{24}$$

$$\frac{\partial u}{\partial y}(t_n,x_i,y_j) = \frac{1}{2}\frac{u(t_n,x_i,y_{j+1}) - u(t_n,x_i,y_{j-1})}{\Delta y}, \tag{25}$$

$$\frac{\partial^2 u}{\partial x^2}(t_n,x_i,y_j) = \frac{u(t_n,x_{i+1},y_j) - 2u(t_n,x_i,y_j) + u(t_n,x_{i-1},y_j)}{(\Delta x)^2}, \tag{26}$$

$$\frac{\partial^2 u}{\partial y^2}(t_n,x_i,y_j) = \frac{u(t_n,x_i,y_{j+1}) - 2u(t_n,x_i,y_j) + u(t_n,x_i,y_{j-1})}{(\Delta y)^2}, \tag{27}$$

$$\frac{\partial^2 u}{\partial x \partial y}(t_n,x_i,y_j) = \frac{1}{4}\frac{u(t_n,x_{i+1},y_{j+1}) - u(t_n,x_{i+1},y_{j-1}) - u(t_n,x_{i-1},y_{j+1}) + u(t_n,x_{i-1},y_{j-1})}{(\Delta x)(\Delta y)} \tag{28}$$

In the following, $u(t_n,x_i,y_j)$ is denoted as $u^n_{i,j}$.

Next, it is explained why the ADI method is not applicable to this study. When the cross derivative is absent, the PDE in eq. (16) reduces to

$$\frac{\partial u}{\partial t} = L_x u + L_y u + F_x + F_y \tag{29}$$

where

$$L_x = A_{xx}\frac{\partial^2}{\partial x^2} + B_x\frac{\partial}{\partial x} + C_x, \quad L_y = A_{yy}\frac{\partial^2}{\partial y^2} + B_y\frac{\partial}{\partial y} + C_y. \tag{30}$$

The ADI method inserts an intermediate time $t_{n+\frac{1}{2}}$ between t_n and t_{n+1}. Then, from t_n to $t_{n+\frac{1}{2}}$, the implicit method is applied to x, and the explicit method is applied to y. Next, from $t_{n+\frac{1}{2}}$ to t_{n+1}, the implicit method is applied to y, and the explicit method is applied to x. These procedures are rewritten as

$$\frac{u^{n+1/2}_{i,j} - u^n_{i,j}}{\Delta t/2} = L_x u^{n+1/2}_{i,j} + L_y u^n_{i,j} + F_x, \tag{31}$$

$$\frac{u^{n+1}_{i,j} - u^{n+1/2}_{i,j}}{\Delta t/2} = L_x u^{n+1/2}_{i,j} + L_y u^{n+1}_{i,j} + F_y. \tag{32}$$

To apply the ADI method, the derivative terms of x and those of y should be decoupled. Therefore, the ADI method is not capable of solving a PDE with cross derivatives.

The fractional step method developed by Yanenko (1971), also known as the splitting method or the locally one-dimensional method, is applied to solve a PDE with cross derivative terms. A fractional step method with the θ scheme rewrites the PDE in eq. (16) as

$$\frac{u_{i,j}^{n+1/2}-u_{i,j}^{n}}{\Delta t}=A_{xx}\frac{\partial^2}{\partial x^2}\left((1-\theta)u_{i,j}^{n}+\theta u_{i,j}^{n+1/2}\right)+A_{xy}\frac{\partial^2}{\partial x\partial y}u_{i,j}^{n}+B_x\frac{\partial}{\partial x}\left((1-\theta)u_{i,j}^{n}+\theta u_{i,j}^{n+1/2}\right)+C_x\left((1-\theta)u_{i,j}^{n}+\theta u_{i,j}^{n+1/2}\right)+F_x, \quad (33)$$

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n+1/2}}{\Delta t}=A_{yy}\frac{\partial^2}{\partial y^2}\left((1-\theta)u_{i,j}^{n+1/2}+\theta u_{i,j}^{n+1}\right)+A_{xy}\frac{\partial^2}{\partial x\partial y}u_{i,j}^{n+1/2}+B_y\frac{\partial}{\partial y}\left((1-\theta)u_{i,j}^{n+1/2}+\theta u_{i,j}^{n+1}\right)+C_y\left((1-\theta)u_{i,j}^{n+1/2}+\theta u_{i,j}^{n+1}\right)+F_y. \quad (34)$$

The fully implicit scheme corresponds to $\theta=1$, and the Crank–Nicolson scheme is obtained by setting $\theta=\frac{1}{2}$. Expressing the differential terms in eq. (33) by finite differences, it can be rewritten as

$$a_{i,j}^{x}u_{i-1,j}^{n+1/2}+b_{i,j}^{x}u_{i,j}^{n+1/2}+c_{i,j}^{x}u_{i+1,j}^{n+1/2}=d_{i,j}^{x}u_{i-1,j}^{n}+e_{i,j}^{x}u_{i,j}^{n}+f_{i,j}^{x}u_{i+1,j}^{n}+g_{i,j}^{x}, \quad (35)$$

where

$$a_{i,j}^{x}=\theta\left(-\frac{(A_{xx})_{i,j}}{(\Delta x)^2}+\frac{(B_x)_{i,j}}{2(\Delta x)}\right), \quad (36)$$

$$b_{i,j}^{x}=\left(\frac{1}{(\Delta\tau)}+\theta\left(\frac{2(A_{xx})_{i,j}}{(\Delta x)^2}-(C_x)_{i,j}\right)\right), \quad (37)$$

$$c_{i,j}^{x}=\theta\left(-\frac{(A_{xx})_{i,j}}{(\Delta x)^2}-\frac{(B_x)_{i,j}}{2(\Delta x)}\right), \quad (38)$$

$$d_{i,j}^{x}=(1-\theta)\left(\frac{(A_{xx})_{i,j}}{(\Delta x)^2}-\frac{(B_x)_{i,j}}{2(\Delta x)}\right), \quad (39)$$

$$e_{i,j}^{x}=\left(\frac{1}{(\Delta\tau)}+(1-\theta)\left(-\frac{2(A_{xx})_{i,j}}{(\Delta x)^2}+(C_x)_{i,j}\right)\right), \quad (40)$$

$$f_{i,j}^{x}=(1-\theta)\left(\frac{(A_{xx})_{i,j}}{(\Delta x)^2}+\frac{(B_x)_{i,j}}{2(\Delta x)}\right), \quad (41)$$

$$g_{i,j}^{x}=(A_{xy})_{i,j}\left(\frac{u_{i+1,j+1}^{n}-u_{i+1,j-1}^{n}-u_{i-1,j+1}^{n}+u_{i-1,j-1}^{n}}{4(\Delta x)(\Delta y)}\right)+(F_x)_i. \quad (42)$$

Similarly, eq. (34) is rewritten as

$$a_{i,j}^{y}u_{i,j-1}^{n+1}+b_{i,j}^{y}u_{i,j}^{n+1}+c_{i,j}^{y}u_{i,j+1}^{n+1}=d_{i,j}^{y}u_{i,j-1}^{n+1/2}+e_{i,j}^{y}u_{i,j}^{n+1/2}+f_{i,j}^{y}u_{i,j+1}^{n+1/2}+g_{i,j}^{y}, \quad (43)$$

where

$$a_{i,j}^{y}=\left(-\frac{(A_{yy})_{i,j}}{(\Delta y)^2}+\frac{(B_y)_{i,j}}{2(\Delta y)}\right), \quad (44)$$

$$b_{i,j}^{y}=\left(\frac{1}{(\Delta\tau)}+\frac{2(A_{yy})_{i,j}}{(\Delta y)^2}-(C_y)_{i,j}\right), \quad (45)$$

$$c_{i,j}^{y}=\left(-\frac{(A_{yy})_{i,j}}{(\Delta y)^2}-\frac{(B_y)_{i,j}}{2(\Delta y)}\right), \quad (46)$$

$$d_{i,j}^{y}=(1-\theta)\left(\frac{(A_{yy})_{i,j}}{(\Delta y)^2}-\frac{(B_y)_{i,j}}{2(\Delta y)}\right), \quad (47)$$

$$e_{i,j}^{y}=\left(\frac{1}{(\Delta\tau)}+(1-\theta)\left(-\frac{2(A_{yy})_{i,j}}{(\Delta y)^2}+(C_y)_{i,j}\right)\right), \quad (48)$$

$$f_{i,j}^{y}=(1-\theta)\left(\frac{(A_{yy})_{i,j}}{(\Delta y)^2}+\frac{(B_y)_{i,j}}{2(\Delta y)}\right), \quad (49)$$

$$g_{i,j}^{y}=(A_{xy})_{i,j}\left(\frac{u_{i+1,j+1}^{n+1/2}-u_{i+1,j-1}^{n+1/2}-u_{i-1,j+1}^{n+1/2}+u_{i-1,j-1}^{n+1/2}}{4(\Delta x)(\Delta y)}\right)+(F_y)_j. \quad (50)$$

Equations (35) and (43) are successively solved from time t_1 to t_N given an initial condition.

4 Empirical study

4.1 Data

Refinitiv Datastream provides spot rates and CDS spreads as follows: spot rates in USD and Euros; CDS spreads for the U.S., Germany, and Portugal quoted in USD and Euros. Previous studies, e.g., Brigo and Alfonsi (2005) and Pan and Singleton (2008), estimated the default intensity process in low CDS spread periods, this study investigates the default intensity process in a financial turmoil. The model is calibrated to the market data on May 31, 2012, at which European financial market experienced the European debt crisis and CDS spreads take extremely high level, and it is interesting to study whether the model is consistent with the market data. Maturities of CDS spreads are 1, 2, 3, 4, 5, 7 and 10 years. Maturities of bond yields are chosen to the same maturities as the CDS.

4.2 Calibration procedure

The stochastic processes for the risk-free interest rate and the default intensity under the martingale measure, $\mathbb{Q}$, are calibrated to the market data on May 31, 2012. The sample date is taken from the mid European debt crisis where CDS spreads are extremely high. The model parameters are chosen so that the mean squared errors of government bond yields and sovereign CDS spreads are minimized. The German government bonds and U.S. Treasury bonds are regarded as the lowest credit risk instruments. Thus, this study investigates the risk-free interest rates in USD and Euros. Corresponding to the currencies for the risk-free interest rates, U.S. and German CDSs are studied. In addition to these CDSs, the Portuguese CDS is chosen to study because the Portuguese CDS spreads skyrocketed in the European debt crisis. The stochastic process for the risk-free interest rate in USD and Euros and the default intensity for the U.S., Germany, and Portugal are calibrated to government bond yields in USD and Euros, and sovereign CDS spreads quoted in USD and Euros at maturities, 1, 2, 3, 4, 5, 7 and 10 years on May 31, 2012.[2]

The difficulties in calibration are concretely described as follows. The German government bond yields depend on the risk-free interest rates in Euros, r_{EUR}, the default intensity for Germany, λ_{DE}, and the correlation between r_{EUR} and λ_{DE}, $\rho_{\mathrm{EUR,DE}}$. Likewise, the U.S. Treasury yields depend on the risk-free interest rates in USD, r_{USD}, the default intensity for the U.S., λ_{US}, and the correlation between r_{USD} and λ_{US}, $\rho_{\mathrm{USD,US}}$. The German CDS spreads quoted in USD are functions of the risk-free interest rates in USD, r_{USD}, the default intensity for German, λ_{DE}, and the correlation between r_{USD} and λ_{DE}, $\rho_{\mathrm{USD,DE}}$. Similarly, the German CDS spreads quoted in Euros are functions of the risk-free interest rates in Euros, r_{EUR}, the default intensity for German, λ_{DE}, and the correlation between r_{EUR} and λ_{DE}, $\rho_{\mathrm{EUR,DE}}$. The same arguments apply to the US CDS spreads quoted in USD and Euros, and Portuguese CDS spreads quoted in USD and Euros. Thus, the calibration procedures are nested, and the number of parameters is enormous (stochastic processes of risk-free interest rates in each currency, default intensities, and correlations between the risk-free interest rates and the default intensities). A joint calibration of r_{USD}, r_{EUR}, λ_{US}, λ_{DE}, λ_{PT}, $\rho_{\mathrm{USD,US}}$, $\rho_{\mathrm{USD,DE}}$, $\rho_{\mathrm{USD,PT}}$, $\rho_{\mathrm{EUR,US}}$, $\rho_{\mathrm{EUR,DE}}$ and $\rho_{\mathrm{EUR,PT}}$ is not an easy task.

To alleviate the calibration burden, a new calibration procedure is proposed. Theoretically, the default intensity calibrated to the CDS spreads quoted in a domestic currency is the same as the default intensity calibrated to the CDS spreads quoted in foreign currencies. Thus, the default intensities are calibrated to the CDS spreads quoted in a domestic currency, and subsequently the CDS spreads in a foreign currency are evaluated using the estimated default intensity. The calibration procedure is concretely described as follows. First, the risk-free interest rate in Euros, r_{EUR}, the default intensity for Germany, λ_{DE}, and the correlation between the risk-free interest rate in Euros and the default intensity for Germany, $\rho_{\mathrm{EUR,DE}}$, are jointly calibrated to the German government bond yields and the German CDS spreads quoted in Euros by minimizing the squared errors of German government bond yields and the sovereign CDS spreads quoted in Euros. Second, the risk-free interest rate in USD, r_{USD}, the default intensity for the U.S., λ_{US}, and the correlation between the risk-free interest rate in USD and the default intensity for the U.S., $\rho_{\mathrm{USD,US}}$, are jointly calibrated to the U.S. Treasury bond yields and the U.S. CDS spreads quoted in USD by minimizing squared errors of the U.S. government bond yields and the sovereign CDS spreads quoted in USD. Third, the correlation between the risk-free interest rate in USD and the default intensity for Germany, $\rho_{\mathrm{USD,DE}}$, is calibrated to the German CDS spreads quoted in USD using the estimated parameters of the risk-free interest rate in USD and the default intensity for Germany by minimizing squared errors of the German CDS spreads quoted in USD. Fourth, likewise, the correlation between the risk-free interest rate in Euros and the default intensity for the U.S.,

[2] Because of a computationally heavy burden, the model is calibrated to a market data on a single date. Matlab program takes a week to estimate all model parameters using a laptop workstation equipped with Intel® Xeron® E-2176M CPU @2.70GHz and 64GB memory.

$\rho_{EUR,US}$, is calibrated to the U.S. CDS spreads quoted in Euros using the estimated parameters of the risk-free interest rate in USD and the default intensity for the U.S. by minimizing squared errors of the U.S. CDS spreads quoted in Euros. Fifth, default intensity for Portugal, λ_{PT}, the correlation between the risk-free interest rate in USD and the default intensity for Portugal, $\rho_{USD,PT}$, and the correlation between the risk-free interest rate in Euros and the default intensity for Portugal, $\rho_{EUR,PT}$, are jointly calibrated to the Portuguese CDS spreads quoted in USD and Euros using the obtained parameters of the risk-free interest rates in USD and Euros by minimizing squared errors of the Portugal CDS spreads quoted in USD and Euros. These sequential calibration procedures enable a reduction in the number of parameters at each step of calibration.

To calibrate the model parameters, the fractional step methods are applied to calculate theoretical values of the government bond yields and the sovereign CDS spread because the fractional step methods can solve PDEs involving cross derivatives within a short calculation time.[3]

4.3 Empirical Results

The calibration results on the model parameters are shown in Table 1 and Table 2. The model parameters of the stochastic processes are given in Table 1. First, the estimated parameters for the risk-free interest rates are examined. Small values of κ for the risk-free interest rate in USD and Euros show a weak mean-reverting property. Negative values of κ for USD and Euros suggest compulsive processes of the risk-free interest rate; however, note that the stochastic processes are estimated under the martingale measure $\mathbb{Q}$, not the physical measure $\mathbb{P}$ and the compulsive properties do not matter. The initial values of x_0 are quite low for the risk-free interest rate in USD and Euros, reflecting the interest rate policy of the Federal Reserve Board and the ECB. Volatilities of the risk-free interest rate in USD are comparable to those in Euros. Next, the calibrated parameters for the default intensities are examined. The strength of mean-reverting, κ, is positive for the U.S. and Germany, while it is negative for Portugal. Again, negative κ is not a problem because the stochastic process is estimated under the martingale measure $\mathbb{Q}$. Volatilities of the default intensity for the U.S. and Germany are comparable, while volatility is quite large for the default intensity for Portugal. The initial value of y_0 is small for the U.S. and Germany, while it is very high for Portugal. The large values of volatility and initial value for Portugal are consistent with the high CDS spreads for Portugal.

Correlations between the risk-free interest rate and the default intensity are presented in Table 2. The correlation between the risk-free rate and the default intensity in the same currency is weak and it takes 0.261867 in USD and −0.253616 in Euro, respectively. On the contrary, the correlation between the risk-free

Table 1

Estimated parameters of the risk-free interest rate and the default intensity

	κ	θ	σ	x_0, y_0
the risk-free interest rate				
x_t (USD)	−0.027058	−0.081485	0.009373	0.001756
x_t (Euro)	−0.010368	−0.118682	0.009331	0.004795
the default intensity				
y_t (U.S.)	0.036886	−4.955170	0.766800	−6.116498
y_t (Germany)	0.164142	−5.017479	0.899207	−6.031938
y_t (Portugal)	−0.022390	−4.112213	3.613508	−3.586914

[3] In addition to the fractional step methods, Monte Carlo simulations are also applied to calculate the government bond yields and the sovereign CDS spreads using the estimated model parameters. The numerical results from the fractional step methods are very close to those from the Monte Carlo simulations. Thus, the calculation results from the fractional step methods are verified.

rate and the default intensity in the foreign currency is close to −1; the correlation between the risk-free rate in USD and the German default intensity is −0.954982 while the correlation between the risk-free rate in Euros and the U.S. default intensity is −0.916590. Stronger negative correlation between the risk-free rate and the default intensity makes CDS spreads higher, and it explain the quanto CDS differentials to some extent.

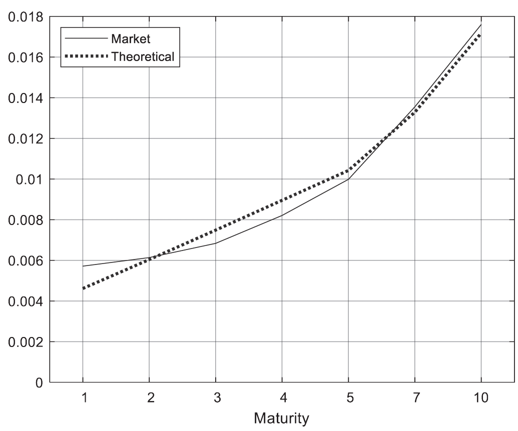
Table 2
Estimated correlations between the risk-free interest rate and the default intensity

	y_t (U.S.)	y_t (Germany)	y_t (Portugal)
x_t (USD)	0.261867	−0.954982	−0.984521
x_t (Euro)	−0.916590	−0.253616	0.989021

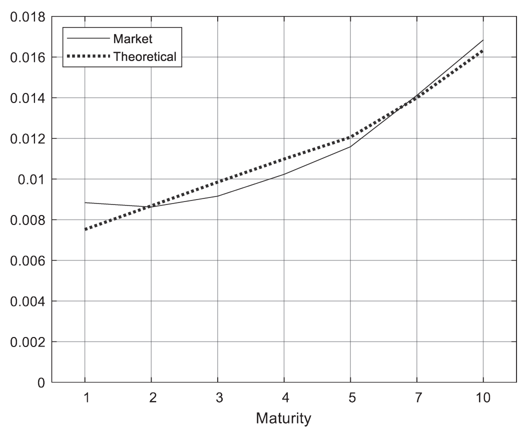
Theoretical spot rates calculated from the calibrated parameters are shown in Figure 1. Casual inspection of Figure 1 makes it clear that the theoretical spot rates are close to the corresponding market spot rates for both USD and Euros.

Theoretical CDS spreads calculated from the calibrated parameters are provided in Figure 2. Casual inspection of Figure 2 makes it clear that the CDS spreads for the U.S. quoted in USD and those for Germany quoted in USD are close to the corresponding market CDS spreads. On the contrary, the theoretical CDS spreads for the U.S. quoted in Euros and those for Germany quoted in USD show poor fit; theoretical CDS spreads for the U.S. quoted in Euros is smaller than the corresponding market spreads by approximately 23% on average; theoretical CDS spreads for Germany quoted in USD are smaller than the corresponding market spreads by approximately 44% on average. Thus, the model fails to explain the quanto CDS spread differentials for the U.S. and Germany. As will be shown in Figure 3 for the case of Portuguese CDS, the stronger negative correlation between the risk-free interest rate and the default intensity is, the higher CDS spread is calculated; however, the correlation does not exceed −1 and it's not possible to obtain a theoretical CDS spread that coincides to the corresponding market CDS spreads.

Thus far, this study investigates the default intensities for the lowest credit risk countries, the U.S. and Germany. Next, the same approach is applied to a valuation of CDS for a high credit risk country. The default intensity for Portugal is calibrated to Portuguese CDS spreads quoted in both USD and Euros using the calibrated parameters for the risk-free interest rates in USD and Euros. The sum of squared pricing errors for Portuguese CDS spreads quoted in USD and those in Euros is minimized. The calibrated parameters are also given in Table 1 and Table 2. Theoretical CDS spreads calculated using the calibrated parameters and the market spreads are depicted in Figure 2. Both the theoretical CDS spreads quoted in USD and Euros are close to the corresponding market spreads. Thus, the model

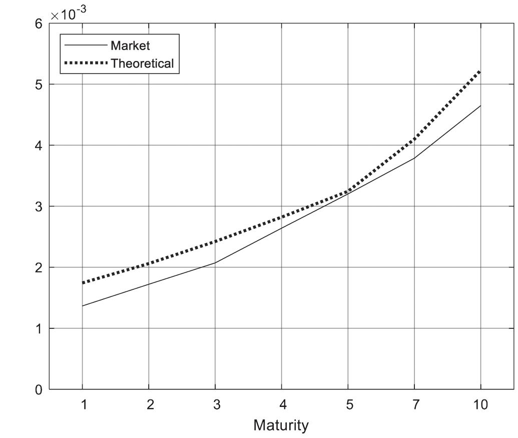


(a) Spot rates in USD

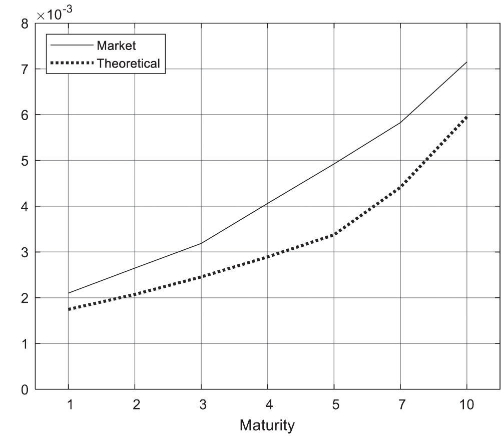


(b) Spot rates in Euros

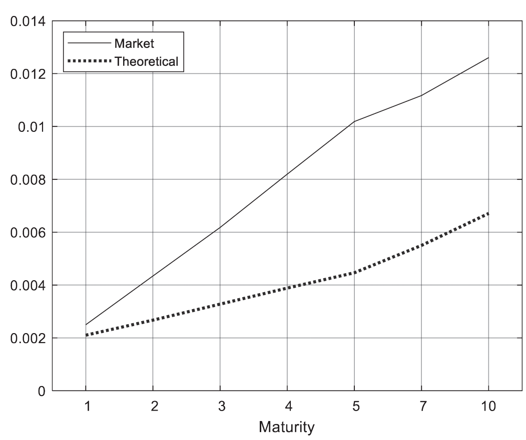
Figure 1. Theoretical and market spot rates quoted in USD (left) and Euros (right).
These figures demonstrate spot rates quoted in USD and Euros. The x-axis is maturity in year and the y-axis is spot rates expressed in decimal.



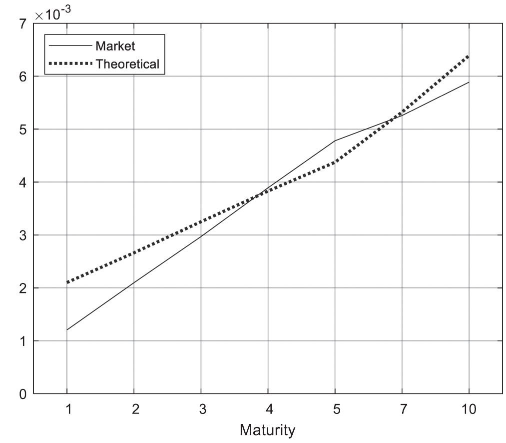
(a) U.S. CDS spreads quoted in USD



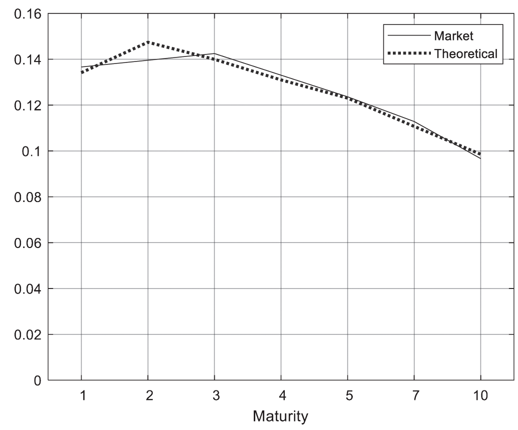
(b) U.S. CDS spreads quoted in Euros



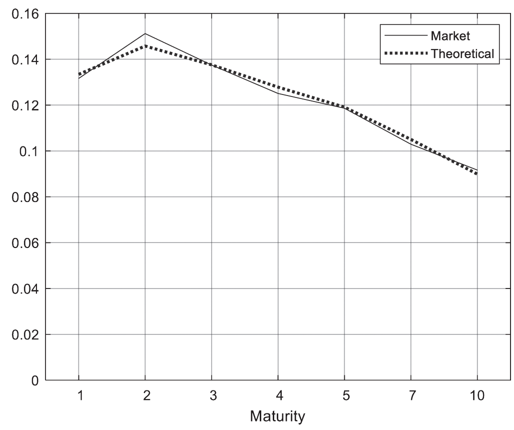
(c) German CDS spreads quoted in USD



(d) German CDS spreads quoted in Euros



(e) Portuguese CDS spreads quoted in USD

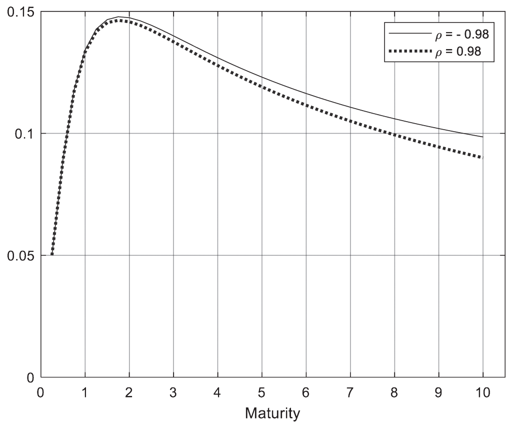


(f) Portuguese CDS spreads quoted in Euros

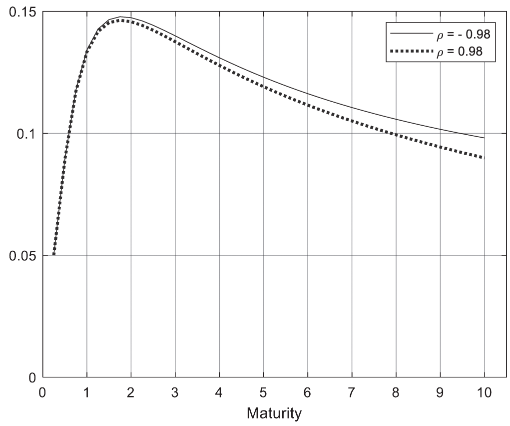
Figure 2. Theoretical and market CDS spreads for the U.S., Germany, and Portugal.

These figures demonstrate CDS spreads for the U.S., Germany, and Portugal quoted in USD and Euros. The x-axis is CDS maturity in year and the y-axis is CDS spreads expressed in decimal.

explains the quanto CDS spread differentials for Portugal. The theoretical CDS spreads for Portugal quoted that both currencies are close to the corresponding market CDS spreads. Note that the correlation between the risk-free interest rate in USD and the default intensity for Portugal is −0.984521 and that between the risk-free rate in Euros and the default intensity for Portugal is 0.989021. The correlation of Portuguese default intensity with spot rates in USD is quite opposite to that in Euros.



(a) Portuguese CDS quoted in USD



(b) Portuguese CDS quoted in Euros

Figure 3. Portuguese CDS spreads quoted in USD (left) and Euros (right) for positive and negative correlations.

The x-axis is CDS maturity in year and the y-axis is CDS spreads expressed in decimal.

4.4 Impact of the Correlation on CDS Spreads

The impact of the correlation between the risk-free interest rate and the default intensity on CDS spreads is examined using calibrated parameters for Portuguese CDS spreads. Values of ρ are set to −0.98 (strong negative correlation) and 0.98 (strong positive correlation), where the former is close to the calibrated value of $\rho_{\mathrm{USD,PT}}$, and the latter is close to the calibrated value of $\rho_{\mathrm{EUR,PT}}$. The parameters concerning the risk-free interest rate and the default intensity for Portugal are set to the estimated values. Portuguese CDS spreads quoted in USD and Euros are depicted in Figure 3. The calculated CDS spreads quoted in USD and Euros are very similar to each other because the market CDS spreads quoted in USD are very close to those quoted in Euros. The CDS spreads change in accordance with the level of the correlation. The difference in CDS spreads at 10-year maturity for the two values of ρ is approximately 9%. As a result, it is concluded that the correlation between the risk-free interest rate and the sovereign default intensity affects CDS spreads, which matters in contrast to previous studies.

5 Conclusion

This study develops a unified model for the risk-free interest rate and the default intensity. The unified model is applied to the valuation of government bonds and sovereign CDS. The model parameters are calibrated to the market data of the government bond yields and the sovereign CDS spreads. The model is calibrated to the market data in the mid-European debt crisis and applied to the valuation of the German, U.S. and Portuguese CDS quoted in USD and Euros. A set of parameters for the unified model explains the government bond yields and the sovereign CDS spreads in a domestic currency. The unified model reproduces the theoretical Portuguese CDS spreads quoted in USD and Euros close to the corresponding market CDS spreads, while it fails to reproduce the market CDS spreads for the German CDS quoted in USD and the U.S. CDS quoted in Euros. Thus, CDS quanto differentials can be partially explained by the unified model by introducing the correlation between the risk-free interest rate and the default intensity. This study assumes the Vasicek model for the risk-free interest rate and the extended Vasicek model for default intensities. Future research applies other stochastic processes for the risk-free interest rate and the default intensity to evaluate government bond yields and sovereign CDS spreads.

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